Vector Geometry Review

Vector Basics - Sections 12.1 and 12.2

Vectors, Components and Representation
Vectors, Scalar Multiplication
Vectors, Addition
Vectors, Magnitude
Vectors, Basis

Dot and Cross Products - Sections 12.3 and 12.4

Vectors, The Dot Product
Vectors, The Cross Product

Lines and Planes - Sections 12.2 and 12.5

Lines in 3-Space
Planes in Space

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1 Vector Basics - Sections 12.1 and 12.2

Vectors

A **vector** is a geometric object that has magnitude (length) and direction.

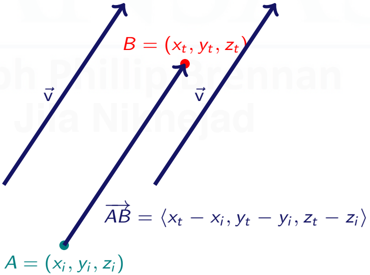
A **scalar** is a constant in $\mathbb{R}$ which has no direction, only magnitude.

Familiar examples of vectors: force, velocity, acceleration, pressure, flux

A vector can be represented geometrically by an arrow AB from A (the initial point) to B (the terminal point). Notation: $\vec{v} = \vec{v} = \overrightarrow{AB}$.

Translating a vector does **not** change it, since the magnitude and direction remain the same.

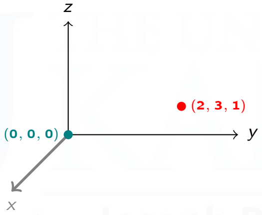
These three arrows all represent the same vector!



▸ Video

Cartesian Representation of Vectors

- Draw a vector $\vec{v}$ with its *initial point* at the origin O.
- The **components** of $\vec{v}$ are the coordinates of the *terminal point* P.



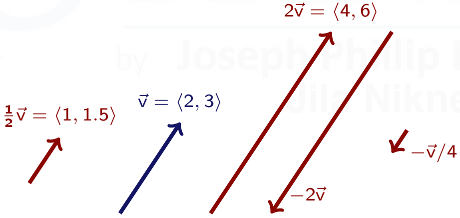
Here $\vec{v} = \overrightarrow{OP} = \langle a, b, c \rangle$.

In general, if $\vec{v} = \overrightarrow{AB}$ where $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ then

$$\vec{v} = \langle\, x_2 - x_1 \;,\; y_2 - y_1 \;,\; z_2 - z_1 \rangle.$$

Scalar Multiplication

- Multiplying a vector $\vec{v}$ by a positive scalar c does not change its direction, but multiplies its magnitude by c.
- If $c < 0$, the direction of $\vec{v}$ is reversed and the magnitude is multiplied by $|c|$.
- Two nonzero vectors $\vec{v}$ and $\vec{w}$ are **parallel** if they are scalar multiples of each other (there exists a scalar c such that $\vec{v} = c\vec{w}$).



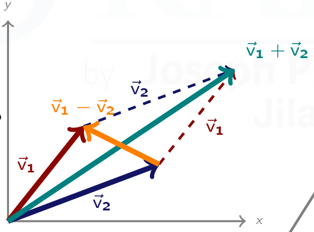
Addition and Subtraction of Vectors

- **Algebraically**, two vectors
  can be added or subtracted by adding or subtracting their components.

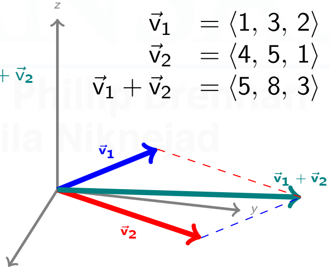
2d	3d
$\langle a,\, b\rangle \pm \langle c,\, d\rangle = \langle a \pm c,\, b \pm d\rangle$	$\langle a,\, b,\, c\rangle \pm \langle p,\, d,\, q\rangle = \langle a \pm p,\, b \pm d,\, c \pm q\rangle$

- **Geometrically**, adding two vectors can be visualized in terms of a parallelogram.

$$
\begin{aligned}
\vec{v}_1 &= \langle 4,\, 5\rangle \\
\vec{v}_2 &= \langle 8,\, 3\rangle \\
\vec{v}_1 + \vec{v}_2 &= \langle 12,\, 8\rangle \\
\vec{v}_1 - \vec{v}_2 &= \langle -4,\, 2\rangle
\end{aligned}
$$



$$
\begin{aligned}
\vec{v}_1 &= \langle 1,\, 3,\, 2\rangle \\
\vec{v}_2 &= \langle 4,\, 5,\, 1\rangle \\
\vec{v}_1 + \vec{v}_2 &= \langle 5,\, 8,\, 3\rangle
\end{aligned}
$$

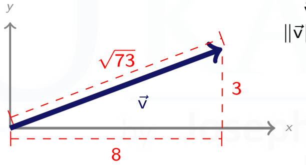


Vector Magnitude

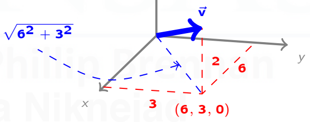
The **magnitude** (or **length**) of a vector $\vec{v}$ is the distance between its initial point and terminal point:

$$\vec{v} = \langle a, b \rangle \qquad \|\vec{v}\| = \sqrt{a^2 + b^2}$$
$$\vec{w} = \langle a, b, c \rangle \qquad \|\vec{w}\| = \sqrt{a^2 + b^2 + c^2}$$

$$\begin{aligned} \vec{v} &= \langle 8,\ 3 \rangle \\ \|\vec{v}\| &= \sqrt{8^2 + 3^2} \\ &= \sqrt{73} \end{aligned}$$



$$\begin{aligned} \vec{v} &= \langle 6,\ 3,\ 2 \rangle \\ \|\vec{v}\| &= \sqrt{6^2 + 3^2 + 2^2} \\ &= 7 \end{aligned}$$



If $\vec{v} = \overrightarrow{AB}$ with $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, then

$$\|\vec{v}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

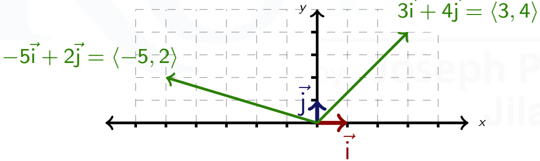
(Note: This is just the usual distance formula.)

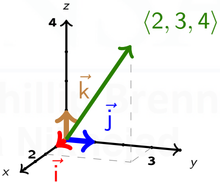
Special Vectors

- The **zero vector** is $\vec{0} = \langle 0, 0 \rangle$ or $\langle 0, 0, 0 \rangle$.

  The zero vector is the **only** vector with magnitude zero. Its direction is undefined.

- **Standard basis vectors** in $\mathbb{R}^2$: $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$
- **Standard basis vectors** in $\mathbb{R}^3$: $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$





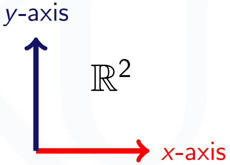
- A **unit vector** is a vector of magnitude one.

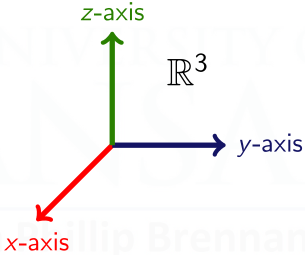
  Unit vectors useful for specifying directions without magnitudes.
  A unit vector in the direction of a given vector can be obtained by multiplying the vector by reciprocal of the magnitude. $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$

  The unit vector in direction $\langle 3, 4 \rangle$ is $\langle \frac{3}{5}, \frac{4}{5} \rangle$.

Cartesian Coordinates in $\mathbb{R}^2$ and $\mathbb{R}^3$

Coordinates represent geometric objects in space by ordered pairs/triples of numbers, so that we can study them with algebra and calculus





- Reference point: the origin O
- Two coordinate axes
- One plane
- Four quadrants

- Reference point: the origin O
- Three coordinate axes
- Three coordinate planes
- Eight octants Link

2 Dot and Cross Products - Sections 12.3 and 12.4

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Dot and Cross Products

In addition to vector addition and scalar multiplication, there are two other important operations on vectors.

1. The **dot product**, which takes two vectors $\vec{v}$ and $\vec{w}$ (either both in $\mathbb{R}^2$ or both in $\mathbb{R}^3$) and produces a *scalar* $\vec{v} \cdot \vec{w}$.

2. The **cross product**, which takes two vectors $\vec{v}$ and $\vec{w}$ (both in $\mathbb{R}^3$) and produces a *vector* $\vec{v} \times \vec{w}$.

It is very important to understand the **geometry** behind the dot and cross product, not just their formulas.

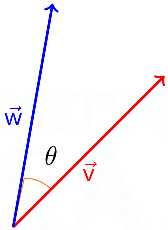
▸ Video

The Dot Product

The **dot product** of two vectors $\vec{v} = \langle a_1, b_1, c_1 \rangle$ and $\vec{w} = \langle a_2, b_2, c_2 \rangle$ is the scalar

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \, \cos(\theta)$$

where θ is the angle between the vectors $\vec{v}$ and $\vec{w}$.



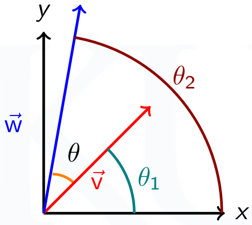
- If θ is acute $(0 \leq \theta < \frac{\pi}{2})$ then $\vec{v} \cdot \vec{w} > 0$.
- If $\vec{v}, \vec{w}$ are orthogonal $(\theta = \frac{\pi}{2})$ then $\vec{v} \cdot \vec{w} = 0$.
- If θ is obtuse $(\frac{\pi}{2} < \theta \leq \pi)$ then $\vec{v} \cdot \vec{w} < 0$.

- The angle between $\vec{v}$ and $\vec{w}$ is $\arccos\left(\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right)$.

The Formula for the Dot Product

Formula in $\mathbb{R}^2$: $\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2$

Formula in $\mathbb{R}^3$: $\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2 + c_1 c_2$



$$a_1 = \|\vec{v}\| \cos(\theta_1) \qquad b_1 = \|\vec{v}\| \sin(\theta_1)$$
$$a_2 = \|\vec{w}\| \cos(\theta_2) \qquad b_2 = \|\vec{w}\| \sin(\theta_2)$$

$$\begin{aligned} a_1 a_2 + b_1 b_2 &= \|\vec{v}\| \, \|\vec{w}\| \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) \right) \\ &= \|\vec{v}\| \, \|\vec{w}\| \, \cos(\theta_2 - \theta_1) \\ &= \|\vec{v}\| \, \|\vec{w}\| \, \cos(\theta) \\ &= \vec{v} \cdot \vec{w}. \end{aligned}$$

The Cross Product

The **cross product** of vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^3$ is the vector

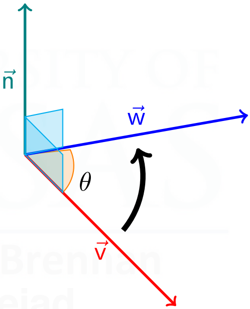
$$\vec{v} \times \vec{w} = \big(\|\vec{v}\| \, \|\vec{w}\| \, \sin(\theta)\big) \, \vec{n}$$

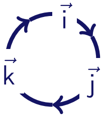
where:

(i) θ is the angle between $\vec{v}$ and $\vec{w}$;

(ii) $\vec{n}$ is the unit vector perpendicular to both $\vec{v}$ and $\vec{w}$, given by the Right-Hand Rule.

(Point the fingers of your right hand toward $\vec{v}$ and then curl them toward $\vec{w}$. Your thumb will point in the direction of $\vec{n}$.)

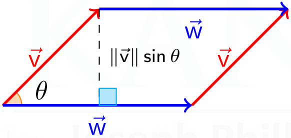




$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$

Properties of the Cross Product

- If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.
- $(\vec{v} \times \vec{w}) \perp \vec{v}$ and $(\vec{v} \times \vec{w}) \perp \vec{w}$.
- $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
- $\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$.



- To calculate the cross product of two vectors in $\mathbb{R}^2$, treat them as vectors in $\mathbb{R}^3$:

$$\vec{v} = \langle v_1, v_2 \rangle = \langle v_1,\ v_2,\ 0 \rangle \qquad \vec{w} = \langle w_1, w_2 \rangle = \langle w_1,\ w_2,\ 0 \rangle$$

In this case $\vec{v} \times \vec{w}$ will always be a multiple of $\vec{k} = \langle 0, 0, 1 \rangle$.

Calculating Cross Products with Determinants

The determinant of a 2×2 matrix is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant of a 3×3 matrix can be calculated by decomposing into a linear combination of 2×2 matrices.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product Formula: $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

3 Lines and Planes - Sections 12.2 and 12.5

Lines in 2-Space (Review)

A line in $\mathbb{R}^2$ is the set of points satisfying a linear equation in x and y.

Point-slope form: The line through (x_0, y_0) with slope m is defined by

$$y - y_0 = m(x - x_0).$$

Slope-intercept form: The line with slope m and y-intercept b is defined by

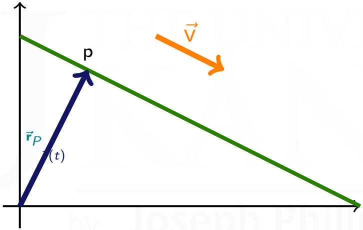
$$y = mx + b.$$

(Exception: A vertical line has undefined slope and cannot be written in either of these forms; its equation is $x = a$.)

Video

Lines in 2-Space: Vector Forms

A line can also be represented using a **direction vector**.
The idea: specify a **point on the line** and a direction to move in.



- The line $y = -\frac{x}{2} + 5$ has slope $m = -\frac{1}{2}$.
- When the x-value changes by $+2$, the y-value changes by -1.
- That is, the line is parallel to the vector $\vec{v} = \langle 2, -1 \rangle$.

Lines in 2-Space: Parametrization

Every line L in $\mathbb{R}^2$ has a **direction vector** $\vec{v}$:

- For any two points P, Q on L, the vector $\overrightarrow{PQ}$ is parallel to $\vec{v}$.
- That is, there is a scalar t such that $\overrightarrow{PQ} = t\vec{v}$.
- Every nonzero multiple of $\vec{v}$ is also a direction vector for L.

- If P is a point on L, then the line can be described by the function

$$\vec{r}(t) = \vec{r}_P + t\vec{v}.$$

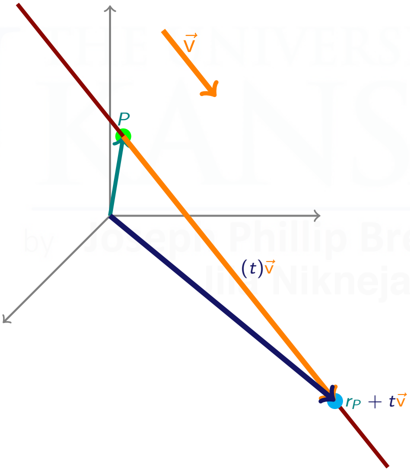
("Start at P, and then change your position by $t\vec{v}$.")

- L has many parametrizations, depending on the choices of P and $\vec{v}$.
  (P is the starting point, t is time, $\vec{v}$ is velocity.)

Lines in 3-Space

Lines in $\mathbb{R}^3$ can be parametrized exactly the same as lines in $\mathbb{R}^2$.

In $\mathbb{R}^3$, a line is still determined by a point and a direction.



Equations of a Line in 3-Space

Let L be a line in $\mathbb{R}^3$, with direction vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$, containing a point $P_0 = (x_0, y_0, z_0)$.

Vector form

$$\vec{r} - \vec{r}_0 = t\vec{v} \text{ for all } t$$

$$\vec{r}(t) = \langle x_0 + tv_1,\ y_0 + tv_2,\ z_0 + tv_3 \rangle$$

Parametric form

$$x = x_0 + tv_1,\ y = y_0 + tv_2,\ z = z_0 + tv_3$$

These two forms are more or less the same.
The name of the parameter t does not matter.

Symmetric form

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3} \qquad (\text{provided } v_1, v_2, v_3 \neq 0)$$

This form consists of <u>two</u> equations on x, y, z, with no parameter.

Lines in $\mathbb{R}^3$: Examples

Example 1: Find equations for the line through point $P = (2, 3, 4)$ parallel to $\vec{v} = \langle 5, 6, 7 \rangle$.

Solution:

Vector form $\vec{r}(t) = \langle 2 + 5t,\ 3 + 6t,\ 4 + 7t \rangle$

Parametric form $x = 2 + 5t \quad y = 3 + 6t \quad z = 4 + 7t$

Symmetric form $\dfrac{x-2}{5} = \dfrac{y-3}{6} = \dfrac{z-4}{7}$

Lines in $\mathbb{R}^3$: Examples

Example 2: Find a vector form of the line through $P = (2, 3, 5)$ and $Q = (4, 2, 1)$.

Solution: The first step is to find a direction vector. Use $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \langle 4-2,\ 2-3,\ 1-5 \rangle = \langle 2,\ -1,\ -4 \rangle.$$

Therefore, a vector form of the line is

$$\vec{r}(t) = \langle 2+2t,\ 3-t,\ 5-4t \rangle.$$

Using the direction vector $\overrightarrow{QP} = \langle -2,\ 1,\ 4 \rangle$ and the point P would give

$$\vec{s}(t) = \langle 2-2t,\ 3+t,\ 5+4t \rangle$$

and starting at Q instead of P would give

$$\vec{q}(t) = \langle 4-2t,\ 2+t,\ 1+4t \rangle.$$

Relative Position of Two Lines in Space

- Two lines can be parallel. Direction vectors for parallel lines are scalar multiples of each other.
- Two non-parallel lines can intersect at a point.
- Two lines can be skew. Skew lines are not parallel and do not intersect.

Link Video

Example 3: The two lines L_1 and L_2 given by the equations Video

$$\begin{array}{llll} L_1: & x = 3 - 2t & y = 1 + t & z = 4 - 3t \\ L_2: & x = -5 + t & y = 4 - t & z = 1 + 6t \end{array}$$

have direction vectors $\vec{v}_1 = \langle -2, 1, -3 \rangle$ and $\vec{v}_2 = \langle 1, -1, 6 \rangle$, which are not scalar multiples — so L_1 and L_2 are not parallel. **Do they intersect?**

Relative Position of Two Lines in Space

Example 3 (continued):

$$\begin{array}{ll} L_1: & \vec{r}_1(t) = \langle 3, 1, 4\rangle + t\langle -2, 1, -3\rangle \\ L_2: & \vec{r}_2(t) = \langle -5, 4, 1\rangle + t\langle 1, -1, 6\rangle \end{array}$$

To check if they intersect, solve the system of equations $\vec{r}_1(t) = \vec{r}_2(s)$:

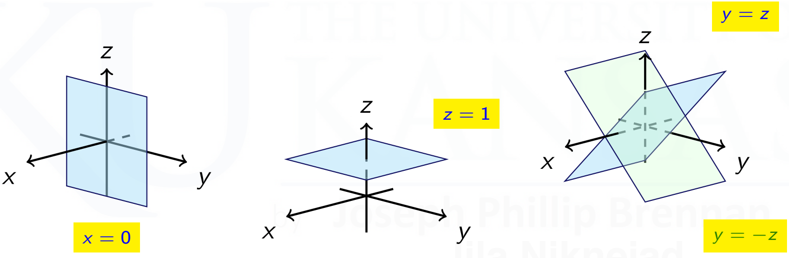
$$\begin{cases} 3 - 2t &= -5 + s \\ 1 + t &= 4 - s \\ 4 - 3t &= 1 + 6s \end{cases}$$

(Be sure to change the name of one of the parameters, since they refer to different lines!)

- Solution: $t = 5$, $s = -2$.
- Lines L_1 and L_2 intersect at $\vec{r}_1(5) = \vec{r}_2(-2) = (-7,\ 6,\ -11)$.
- If the system has no solution, then the lines are skew.

Planes in Space

If a line in $\mathbb{R}^3$ is defined by <u>two</u> linear equations (in its symmetric form), what kind of set is defined by <u>one</u> linear equation? **A plane.**



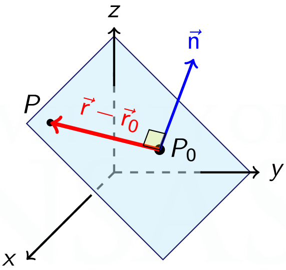
Question: How do we translate between the algebraic equation of a plane and its geometric properties?

▸ Video

Equations for Planes

$P_0(x_0, y_0, z_0)$: point in $\mathbb{R}^3$
$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$
$\vec{n} = \langle n_1, n_2, n_3 \rangle$: nonzero vector

Then there is a unique plane F that passes through P_0 and is orthogonal to $\vec{n}$.



Let $P(x, y, z)$ be a general point on the plane F and let $\vec{r} = \langle x, y, z \rangle$.

Vector equation of F $\qquad \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$
Scalar equation of F $\qquad n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$

The vector $\vec{n}$ is called a **normal vector** to F.
Any nonzero multiple of $\vec{n}$ is also a normal vector to F.

Equations for Planes: Examples

Example 4: Find equations for the plane containing the point $(7, -8, 5)$ with normal vector (i) $\vec{n} = \langle -2, 1, 4 \rangle$; (ii) $\vec{n} = \langle -2, 0, 4 \rangle$; (iii) $\vec{n} = \langle 0, 0, 3 \rangle$.

Solution:

(i)
$$\begin{aligned} & \langle -2, 1, 4 \rangle \cdot \langle x-7,\ y+8,\ z-5 \rangle = 0 \\ \text{or} \quad & -2(x-7) + (y+8) + 4(z-5) = 0 \\ \text{or} \quad & -2x + y + 4z = -2 \end{aligned}$$

(ii)
$$\begin{aligned} & \langle -2, 0, 4 \rangle \cdot \langle x-7,\ y+8,\ z-5 \rangle = 0 \\ \text{or} \quad & -2(x-7) + 4(z-5) = 0 \\ \text{or} \quad & -2x + 4z = 6 \end{aligned}$$

(iii)
$$\begin{aligned} & \langle 0, 0, 3 \rangle \cdot \langle x-7,\ y+8,\ z-5 \rangle = 0 \\ \text{or} \quad & 3(z-5) = 0 \\ \text{or} \quad & z = 5 \end{aligned}$$

Equations for Planes: Examples

Example 5: Find an equation through the plane F containing the three points $A(1, -2, 0)$, $B(3, 1, 4)$, $C(2, 1, -2)$.

Solution: Geometrically, three points certainly determine a plane. So we need a normal vector.

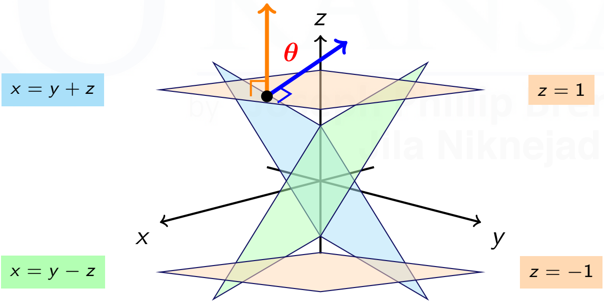
- The vectors $\overrightarrow{AB} = \langle 2, 3, 4 \rangle$ and $\overrightarrow{AC} = \langle 1, 3, -2 \rangle$ both lie **in** F.
- The normal vector $\vec{n}$ needs to be orthogonal to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
- Thus, we can use the **cross product** $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -18, 8, 3 \rangle$ for $\vec{n}$.

One solution: $\boxed{-18(x-1) + 8(y+2) + 3z = 0.}$

There are other possibilities: $-18(x-3) + 8(y-1) + 3(z-4) = 0$, etc.

Relative Position of Two Planes in Space

- Two planes are <u>parallel</u> exactly when their normal vectors are scalar multiples of one another.
- If two planes are not parallel, then they <u>intersect</u>.
  - When two planes intersect, their intersection is a line.
  - The angle θ between two planes is the angle between their normal vectors (at most $\pi/2$). If $\theta = 0$ then the planes are parallel.



Relative Position of Two Planes in Space

Example 6: Determine the line L of intersection of the planes F_1 and F_2 whose equations are

$$F_1: \quad 2x - 3y + 5z = 1, \qquad F_2: \quad 3x - 4y = 7.$$

Solution: Normal vectors for the planes: $\vec{n}_1 = \langle 2, -3, 5 \rangle$, $\vec{n}_2 = \langle 3, -4, 0 \rangle$.

Since L lies in both planes, its direction $\vec{v}$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 20, 15, 1 \rangle.$$

Solve the system $2x - 3y + 5z = 1$, $3x - 4y = 7$ to get a point on L. There are many solutions; one is $(17, 11, 0)$.

Answer: $\boxed{\vec{r}(t) = \langle 17 + 20t,\ 11 + 15t,\ t \rangle.}$ ▸ Video